THE NON-POLYNOMIAL CONSERVATION LAWS AND INTEGRABILITY ANALYSIS OF GENERALIZED RIEMANN TYPE HYDRODYNAMICAL EQUATIONS

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Abstract. Based on the gradient-holonomic algorithm we analyze the integrability property of the generalized hydrodynamical Riemann type equation $D_t^N u = 0$ for arbitrary $N \in \mathbb{Z}_+$. The infinite hierarchies of polynomial and non-polynomial conservation laws, both dispersive and dispersionless are constructed. Special attention is paid to the cases $N = 2, 3$ and $N = 4$, for which the conservation laws, Lax type representations and bi-Hamiltonian structures are analyzed in detail. We also show that the case $N = 2$ is equivalent to a generalized Hunter-Saxton dynamical system, whose integrability follows from the results obtained. As a byproduct of our analysis we demonstrate a new set of non-polynomial conservation laws for the related Hunter-Saxton equation.

1. Introduction

Nonlinear hydrodynamic equations are of constant interest still from classical works by B. Riemann, who had extensively studied them in general three-dimensional case, having paid special attention to their one-dimensional spatial reduction, for which he devised the generalized method of characteristics and Riemann invariants. These methods appeared to be very effective [1, 4, 18] in investigating many types of nonlinear spatially one-dimensional systems of hydrodynamical type and, in particular, the characteristics method in the form of a "reciprocal" transformation of variables has been used recently in studying a so called Gurevich-Zybin system [2, 3] in [8] and a Whitham type system in [18, 9]. Moreover, this method was further effectively applied to studying solutions to a generalized [10] (owing to D. Holm and M. Pavlov) Riemann type hydrodynamical system

$$D_t^N u = 0, \quad D_t := \partial/\partial t + u\partial/\partial x, \quad N \in \mathbb{Z}_+, \tag{1.1}$$

where $dx/dt = u \in C^\infty(\mathbb{R};\mathbb{R})$ is the corresponding characteristic flow velocity along the real axis $\mathbb{R}$.

We will consider, for convenience, the hydrodynamical equation (1.1) on the 2π-periodic space of functions $\mathcal{M}_0 := C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$, which can be, obviously, equivalently rewritten as the following nonlinear dynamical system in the augmented functional manifold $\mathcal{M} := C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^N)$ on the vector $\hat{u} := (u^{(0)} := u, u^{(1)} := D_t u^{(0)}, u^{(2)} := D_t u^{(1)}, ..., u^{(N-1)} := D_t u^{(N-2)})^\intercal \in \mathcal{M}$:

$$\begin{aligned} u_t^{(0)} &= u^{(1)} - u^{(0)} u_x^{(1)}, \\ u_t^{(1)} &= u^{(2)} - u^{(0)} u_x^{(2)}, \\ &...... \\ u_t^{(N-2)} &= u^{(N-1)} - u^{(0)} u_x^{(N-2)} \\ u_t^{(N-1)} &= -u^{(1)} u_x^{(N-1)}. \end{aligned} \tag{1.2}$$

The dynamical system (1.2) possesses a very interesting and important property: the partial flows of the velocity components $u^{(j)}, j = \overline{0, N-1}$, are realized along the axis $\mathbb{R}$ with the same characteristic velocity $dx/dt = u^{(0)}$. This is exactly the case, deeply studied by Riemann (see, for example, [1]) when one can introduce the so-called "Riemann invariants" making it possible to obtain a suitable separation of dependent variables important for the integration. Really, we can

recurrent set of differential equations in full differentials:

$$
\begin{aligned}
du^{(0)} &= u^{(1)}dt,\\
du^{(1)} &= u^{(2)}dt,\\
&\ \ ......\\
du^{(N-2)} &= u^{(N-1)}dt,\\
du^{(N-1)} &= 0,
\end{aligned}
\tag{1.3}
$$

whose solution is easily found by means of simple integration in the parametric form as

$$
\begin{aligned}
u^{(N-1)} &: = z, \quad z = \beta_N(x - \frac{zt^N}{N!} - \sum_{j=1}^{N-1}\frac{t^j}{j!}\beta_{N-j}(z)),\\
u^{(N-2)} &= zt + \beta_1(z), \quad u^{(N-3)} = \frac{zt^2}{2!} + t\beta_1(z) + \beta_2(z),\\
u^{(N-3)} &= \frac{zt^3}{3!} + \frac{t^2}{2!}\beta_1(z),\\
&\ \ ......\\
u^{(1)} &= \frac{zt^{N-2}}{(N-2)!} + \sum_{j=0}^{N-3}\frac{t^j}{j!}\beta_{N-2-j}(z),\\
u^{(0)} &= \frac{zt^{N-1}}{(N-1)!} + \sum_{j=0}^{N-2}\frac{t^j}{j!}\beta_{N-1-j}(z),
\end{aligned}
\tag{1.4}
$$

where $\beta_j \in C^\infty(\mathbb{R};\mathbb{R}), j = \overline{1, N-1}$, and $\beta_N \in C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ are arbitrary smooth functions, depending on a suitable first integral $z \in C^\infty(\mathbb{R}^N;\mathbb{R})$ of the system (1.3). The presented above result coincides in some part with that obtained before by M. Pavlov in [14], and can be used for constructing special solutions in analytical form to the generalized Riemann type hydrodynamical equation (1.1).

As it was stated before in [16, 8, 7, 15, 14, 10] the Riemann type hydrodynamical system (1.2) at $N = 2$ and $N = 3$ possesses additional very interesting properties, being a completely integrable bi-Hamiltonian system. In particular, it possesses infinite hierarchies of dispersionless and dispersive conservation laws, which can have an important hydrodynamical interpretation and may be used for constructing a wide class of other special quasi-periodic and solitonic solutions.

In spite of the exact integrability of dynamical system (1.1) by means of the classical characteristics method, the solutions obtained this way are, to the regret, of very vague usefulness, as they are given in the entangled and involved form not fitting for studying solutions belonging to some specially assigned classes of functions, for instance, fast-decreasing, quasi-periodic and etc. Thereby, additional studying of the mathematical structures associated with dynamical system (1.1) by means of modern symplectic theory techniques is as much as could needed, and what is a topic of our present investigation.

The next section below is devoted to the Hamiltonian analysis of the hydrodynamical system (1.2) at $N = 2$, $N = 3$ and $N = 4$, as well as to the description of their new hierarchies of conservation laws, the related co-symplectic structures and Lax type representations.

2. The generalized Riemann type hydrodynamical equation at N=2: conservation laws, bi-Hamiltonian structure and Lax type representation

Consider the generalized Riemann type hydrodynamical equation (1.1) at $N = 2:$

$$D_t^2 u = 0, \tag{2.1}$$

where $D_t = \partial/\partial t + u\partial/\partial x$, which is equivalent to the following dynamical system:

$$\left.u_t = v - uu_x\ \right\} := K[u, v], \tag{2.2}$$

where $K : \mathcal{M} \to T(\mathcal{M})$ is a related vector field on the 2π-periodic smooth nonsingular functional phase space $\mathcal{M} := \{(u,v)^\intercal \in C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^2) : u_x^2 - 2v_x \neq 0, x \in \mathbb{R}\}$. As we are interested first in the conservation laws for the system (2.2), the following proposition holds.

Proposition 2.1. *Let $H(\lambda) := \int_0^{2\pi} h(x;\lambda)dx \in D(\mathcal{M})$ be an almost everywhere smooth functional on the manifold $\mathcal{M}$, depending parametrically on $\lambda \in \mathbb{C}$, and whose density satisfies the differential condition*

$$h_t = \lambda(uh)_x \tag{2.3}$$

for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ on the solution set of dynamical system (2.2). Then the following iterative differential relationship

$$(f/h)_t = \lambda(uf/h)_x \tag{2.4}$$

holds, if a smooth function $f \in C^\infty(\mathbb{R};\mathbb{R})$ (parametrically depending on $\lambda \in \mathbb{C}$) satisfies for all $t \in \mathbb{R}$ the linear equation

$$f_t = 2\lambda u_x f + \lambda u f_x. \tag{2.5}$$

Proof. We have from (2.3)-(2.5) that

$$\begin{aligned} (f/h)_t &= f_t/h - fh_t/h^2 = f_t/h - \lambda f u_x/h - \lambda f u h_x/h^2 = \\ &= f_t/h + \lambda f u(1/h)_x - \lambda u_x f/h = \\ &= \lambda(uf)_x/h + \lambda u f(1/h)_x = \lambda(uf/h)_x, \end{aligned} \tag{2.6}$$

proving the proposition. □

The obvious generalization of the previous proposition is read as follows.

Proposition 2.2. *If a smooth function $h \in C^\infty(\mathbb{R};\mathbb{R})$ satisfies the relationship*

$$h_t = ku_x h + uh_x, \tag{2.7}$$

where $k \in \mathbb{R}$, then

$$H = \int_0^{2\pi} h^{1/k} dx \tag{2.8}$$

is a conservation law for the Riemann type hydrodynamical system (2.2).

Remark 2.3. Let $\hat{h} \in C^\infty(\mathbb{R};\mathbb{R})$ satisfy the differential relationship $\hat{h}_t = (\hat{h}u)_x$, then $f = \hat{h}^2$ is a solution to equation (2.4).

Remark 2.4. If functions $h_j \in C^\infty(\mathbb{R};\mathbb{R}), j \in \mathbb{Z}_+$, satisfy the relationships $h_{j,t} = \lambda(h_j u)_x, j \in \mathbb{Z}_+$, $\lambda \in \mathbb{C}$, then the functionals

$$H_{(i,j)} = \sum_{n \in \mathbb{Z}_+} k_n^{(i,j)} \int_0^{2\pi} h_i^{2^n} h_j^{(1-2^n)} \tag{2.9}$$

with $k_n^{(i,j)} \in \mathbb{R}$, $n \in \mathbb{Z}_+$, $i,j \in \mathbb{Z}_+$, being arbitrary constants, are conserved quantities to equation (2.2). This formula, in particular, makes it possible to construct an infinite hierarchy of non-polynomial conserved quantities for the Riemann type hydrodynamical system (2.2).

Example 2.5. The following non-polynomial functionals

$$
\begin{aligned}
H_4^{(\frac{1}{3})} &= \int_0^{2\pi} \sqrt{u_x^2 - 2v_x}dx, \qquad H_7^{(\frac{1}{3})} = \int_0^{2\pi} (u_x v_{xx} - u_{xx} v_x)^{1/3}\, dx, \\
H_7^{(\frac{1}{2})} &= \int_0^{2\pi} \sqrt{v(u_x^2 - 2v_x)}dx, \\
H_8^{(\frac{1}{3})} &= \int_0^{2\pi} \left(k_1 u(u_{xx} v_x - u_x v_{xx}) + k_1 v_{xx} v + k_2(u_x^2 v - 2v_x^2)\right)^{1/3} dx, \\
H_9^{(\frac{1}{6})} &= \int_0^{2\pi} (u_{xx} v_{xxx} - u_{xxx} v_{xx})^{\frac{1}{6}}\, dx, \\
H_9^{(\frac{1}{4})} &= \int_0^{2\pi} (u_x(u_{xx} v_x - u_x v_{xx}) + v_{xx} v_x))^{\frac{1}{4}}, \\
H_{10}^{(\frac{1}{6})} &= \int_0^{2\pi} \left(2u_{xx}(u_x v_{xx} - u_{xx} v_x) - v_{xx}^2\right)^{\frac{1}{6}}
\end{aligned} \tag{2.10}
$$

are conservation laws for the Riemann type dynamical system (2.2).

Quite different conservation laws have been obtained in [7, 8] using the recursion operator technique. The corresponding recursion operator proves to generate no new conservation law, if one applies it to the non-polynomial conservations laws (2.10).

We also notice that dynamical system (2.2), as it was shown before in [8, 13], can be transformed via the substitution

$$
v = \frac{1}{2}\partial^{-1}(u_x^2 + \eta^2) \tag{2.11}
$$

into the generalized two-component Hunter - Saxton equation:

$$
\begin{aligned}
u_{x,t} &= -\frac{1}{2}u_x^2 - u u_{xx} + \frac{1}{2}\eta^2, \\
\eta_t &= -(u\eta)_x.
\end{aligned} \tag{2.12}
$$

This equation allows the simple reduction to the Hunter - Saxton dynamical system [5, 15, 13] at $\eta = 0:$

$$
u_{xt} = -\frac{1}{2}u_x^2 - u u_{xx}. \tag{2.13}
$$

The non-polynomial conservation laws (2.10), upon rewriting with respect to the substitution (2.11), give rise to the related non-polynomial conservations laws for the generalized two-component Hunter - Saxton dynamical system (2.12). Moreover, if we further apply the reduction $\eta = 0$, we obtain, respectively, new non-polynomial conservation laws for the Hunter - Saxton dynamical system (2.13), supplementing those found before in [15, 13].

Example 2.6. The following functionals

$$
\begin{aligned}
H_7^{(\frac{1}{3})} &= \int_0^{2\pi} \left(u_{xx} u_x^2\right)^{\frac{1}{3}} dx, \quad H_9^{(\frac{1}{6})} = \int_0^{2\pi} \frac{u_{xxx} u + 2u_{xx} u_x}{\sqrt{u_{xx}}} dx, \\
H_8^{(\frac{1}{3})} &= \int_0^{2\pi} \left[u_{xx} u_x(\partial^{-1} u_x^2) - u_{xx} u_x^2 u\right]^{\frac{1}{3}} dx
\end{aligned} \tag{2.14}
$$

are the conservation laws for the Hunter-Saxton dynamical system (2.13).

All of these and many others non-polynomial conservation laws can be easily obtained using Proposition (2.2). For example, the next functionals

$$
\begin{aligned}
H_{(n,m)} &= \int_0^{2\pi} \left(u_{xx}^n u_x^m\right)^{\frac{2}{m+4n}} dx, \quad H_{(1)} = \int_0^{2\pi} u_x^2(\partial^{-1}u_x^2)^2 dx, \\
H_{(2)} &= \int_0^{2\pi} \sqrt{u_{xx}}dx, \qquad H_{(3)} = \int_0^{2\pi} \sqrt{u_{xx}(\partial^{-1}u_x^2)}dx, \\
H_{(4)} &= \int_0^{2\pi} \left[(\partial^{-1}u_x^2)(uu_xu_{xx}^2 - u_{xx}^2(\partial^{-1}u_x^2))\right]^{\frac{2}{9}} dx
\end{aligned} \tag{2.15}
$$

are also conservation laws for the Hunter-Saxton dynamical system (2.13), where $m \neq -4n$ and $n, m \in \mathbb{Z}$.

Now we proceed to analyzing the Hamiltonian properties of the dynamical system (2.2), for which we will search for solutions to the determining [18, 21] Nöther equation

$$
L_K\vartheta = \vartheta_t - \vartheta K^{\prime,*} - K'\vartheta = 0. \tag{2.16}
$$

where L_K denotes the corresponding Lie derivative on $\mathcal{M}$ subject to the vector field $K : \mathcal{M} \to T(\mathcal{M})$, $K' : T(\mathcal{M}) \to T(\mathcal{M})$ is its Frechet derivative, $K'^{*} : T^*(\mathcal{M}) \to T^*(\mathcal{M})$ is its conjugation with respect to the standard bilinear form $(\cdot,\cdot)$ on $T^*(\mathcal{M})\times T(\mathcal{M})$, and $\vartheta : T^*(\mathcal{M}) \to T(\mathcal{M})$ is a suitable implectic operator on $\mathcal{M}$, with respect to which the following Hamiltonian representation

$$
K = -\vartheta \; grad \; H_\vartheta \tag{2.17}
$$

for some smooth functional $H_\vartheta \in D(\mathcal{M})$ holds. To show this, it is enough to find, for instance by means of the small parameter method [18, 20], a non-symmetric ($\psi' \neq \psi^{\prime,*}$) solution $\psi \in T^*(\mathcal{M})$ to the following Lie-Lax equation:

$$
\psi_t + K^{\prime,*}\psi = grad \; \mathcal{L} \tag{2.18}
$$

for some suitably chosen smooth functional $\mathcal{L} \in \mathcal{D}(M)$. As a result of easy calculations one obtains that

$$
\psi = (v, 0)^\intercal, \quad L = \frac{1}{2}\int_0^{2\pi} v^2 dx. \tag{2.19}
$$

Making use of (2.18) jointly with the classical Legendrian relationship

$$
H_\vartheta := (\psi, K) - \mathcal{L} \tag{2.20}
$$

for the suitable Hamiltonian function, one easily obtains the corresponding symplectic structure

$$
\vartheta^{-1} := \psi' - \psi^{\prime,*} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.21}
$$

and the non-singular Hamilton function

$$
H_\vartheta := \frac{1}{2}\int_0^{2\pi} (v^2 + v_x u^2) dx. \tag{2.22}
$$

Since the operator (2.21) is nonsingular, we obtain the corresponding implectic operator

$$
\vartheta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.23}
$$

necessarily satisfying the Nöther equation (2.16).

It is worth here to observe that the determining Lie-Lax equation (2.18) possesses still other solution

$$
\psi = (\frac{u_x}{2}, -\frac{u_x^2}{2v_x}), \;\; \mathcal{L} = \frac{1}{4}\int_0^{2\pi} uv_x dx, \tag{2.24}
$$

giving rise, owing to expressions (2.21) and (2.20), to the new co-implectic (singular "symplectic") structure

$$
\eta^{-1} := \psi' - \psi^{\prime,*} = \begin{pmatrix} \partial & -\partial u_x v_x^{-1} \\ -u_x v_x^{-1}\partial & u_x^2 v_x^{-2}\partial + \partial u_x^2 v_x^{-2} \end{pmatrix} \tag{2.25}
$$

on the manifold $\mathcal{M}$, subject to which the Hamiltonian functional equals

$$
H_\eta := \frac{1}{2}\int_0^{2\pi} (u_x v - v_x u) dx, \tag{2.26}
$$

supplying the second Hamiltonian representation

$$K = -\eta \; grad \; H_\eta \tag{2.27}$$

of the Riemann type hydrodynamical system (2.2). The co-implectic structure (2.25) is singular, since $\hat{\eta}^{-1}(u_x, v_x)^\intercal = 0$, nonetheless one can calculate its inverse expression

$$\eta := \begin{pmatrix} -\partial^{-1} & u_x\partial^{-1} \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x \end{pmatrix}, \tag{2.28}$$

Moreover, the corresponding implectic structure $\eta : T^*(\mathcal{M}) \to T^*(\mathcal{M})$ satisfies the determining Nöther equation

$$L_K\eta = \eta_t - \eta K^{\prime,*} - K^\prime\eta = 0, \tag{2.29}$$

whose solutions can also be obtained by means of the small parameter method [18, 19]. We remark also that, owing to the general symplectic theory results [18, 19, 20, 21] for nonlinear dynamical systems on smooth functional manifolds, operator (2.25) defines on the manifold $\mathcal{M}$ a closed functional differential two-form. Thereby it is *a priori* co-implectic (in general, singular symplectic), satisfying on $\mathcal{M}$ the standard Jacobi brackets condition.

As a result, the second implectic operator (2.28), being compatible [18, 21] with the implectic operator (2.23), gives rise to a new infinite hierarchy of polynomial conservation laws

$$\gamma_n := \int_0^1 d\lambda < (\vartheta^{-1}\eta)^n grad \; H_\vartheta[u\lambda\}, u > \tag{2.30}$$

for all $n \in \mathbb{Z}_+$. Having defined the recursion operator $\Lambda := \vartheta^{-1}\eta$, one also finds from (2.30), (2.16) and (2.29) that the following Lax type relationship

$$L_K\Lambda = \Lambda_t - [\Lambda, K^{',*}] = 0 \tag{2.31}$$

holds. If to construct now the asymptotical expansion $\varphi(x;\lambda) \simeq \sum_{j\in\mathbb{Z}_+} \lambda^{1-2j} grad \; \gamma_{j-1}[u,v]$ as $\lambda \to \infty$, it is easy to obtain from (2.30) that the gradient like relationship

$$\lambda^2\vartheta\varphi(x;\lambda = \eta\varphi(x;\lambda) \tag{2.32}$$

holds. The latter relationship, making use of the implectic operators (2.23) and (2.28), can be represented in the following two factorized forms:

$$\varphi(x;\lambda) := \begin{pmatrix} \varphi_1(x;\lambda) \\ \varphi_2(x;\lambda) \end{pmatrix} = \begin{pmatrix} -4\lambda^3 f_1^2 + 2\lambda v_x f_2^2 \\ -4\lambda^2 f_1 f_2 - 2\lambda u_x f_2^2 \end{pmatrix} = \begin{pmatrix} -2\lambda(f_1f_2)_x \\ -(f_2^2)_x \end{pmatrix}, \tag{2.33}$$

where a vector $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^2)$ lies in an associated to manifold $\mathcal{M}$ vector bundle $\mathcal{L}(\mathcal{M};\mathbb{E}^2)$, whose fibers are isomorphic to the complex Euclidean vector space $\mathbb{E}^2$. Take now into account [18, 20] that the Lie-Lax equation

$$L_K\varphi(x;\lambda) = d\varphi(x;\lambda)/dt + K^{\prime,*}\varphi(x;\lambda) = 0 \tag{2.34}$$

can be transformed equivalently for all $x, t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ into the following evolution system:

$$D_t\varphi = \begin{pmatrix} 0 & v_x \\ -1 & -u_x \end{pmatrix}\varphi, \quad D_t = \partial/\partial t + u\partial/\partial x. \tag{2.35}$$

The equation (2.35), owing to the relationship (2.32) and the obvious identity

$$D_t f_x + u_x f_x = (D_t f)_x, \tag{2.36}$$

can be further split into the adjoint to (2.35) system

$$D_t f = q(\lambda) f, \quad q(\lambda) := \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix}, \tag{2.37}$$

where a vector $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^2)$ satisfies the following linear equation

$$f_x = \ell[u,v;\lambda] f, \quad \ell[u,v;\lambda] := \begin{pmatrix} -\lambda u_x & -v_x \\ 2\lambda^2 & \lambda \end{pmatrix}, \tag{2.38}$$

compatible with (2.37). Moreover, as a result of (2.37) and (1.4), the general solution to (2.38) allows the following functional representation:

$$
\begin{aligned}
f_1(x,t) &= \tilde{g}_1(u-tv, x-tu+vt^2/2), \\
f_2(x,t) &= -t\lambda\tilde{g}_1(u-tv, x-tu+vt^2/2)+ \\
&\quad +\tilde{g}_2(u-tv, x-tu+vt^2/2),
\end{aligned}
\tag{2.39}
$$

where $\tilde{g}_j \in C^\infty(\mathbb{R}^2;\mathbb{C}), j=\overline{1,2}$, are arbitrary smooth complex valued functions. Now combining together the obtained relationships (2.37) and (2.38), we can formulate the following proposition.

Proposition 2.7. *The Riemann type hydrodynamical system (1.1) is equivalent to a completely integrable bi-Hamiltonian flow on the functional manifold $\mathcal{M}$, allowing the Lax type representation*

$$
\begin{gathered}
f_x = \ell[u,v;\lambda]f, \quad f_t = p(\ell)f, \qquad p(\ell) := -u\ell[u,v;\lambda]+q(\lambda), \\
\ell[u,v;\lambda] := \begin{pmatrix} -\lambda u_x & -v_x \\ 2\lambda^2 & \lambda u_x \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix}, \\
p(\ell) = \begin{pmatrix} \lambda u_x u & v_x u \\ -\lambda-2\lambda^2 u & -\lambda u_x u \end{pmatrix},
\end{gathered}
\tag{2.40}
$$

where $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^2)$ and $\lambda \in \mathbb{C}$ is an arbitrary spectral parameter.

Remark 2.8. It is worth to mention here that equation (2.37) is equivalent on the solution set of the Riemann type hydrodynamical system (2.2) to the alone equation

$$
D_t^2 f_2 = 0 \Longleftrightarrow D_t f_1 = 0, D_t f_2 = -\lambda f_1, \tag{2.41}
$$

where vector $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^2)$ satisfies for all $\lambda \in \mathbb{C}$ the compatibility condition (2.38) and whose general solution is represented in the functional form (2.39).

Concerning the set of conservation laws $\{H_0^{(1/2)}, H_1^{(1/2)}\}$, constructed above, they can be extended to an infinite hierarchy $\{H_j^{(1/2)} \in D(\mathcal{M}) : j \in \mathbb{Z}_+ \cup \{-1\}\}$, where

$$
H_j^{(1/2)} := \int_0^{2\pi} \sigma_{2j+1}[u,v]dx, \tag{2.42}
$$

and the affine generating function $\sigma(x;\lambda) := \frac{d}{dx}\ln f_2(x;\lambda) \simeq \sum_{j=\mathbb{Z}_+\cup\{-1\}} \sigma_j[u,v]\lambda^{-j}$ as $\lambda \to \infty$ satisfies the following functional equation:

$$
(\sigma - \lambda u_x)_x + \sigma^2 + \lambda^2(2v_x - u_x^2) = 0. \tag{2.43}
$$

In addition, the gradient functional $\varphi(x;\lambda) := grad\, \gamma(x;\lambda) \in T^*(\mathcal{M})$, where $\gamma(\lambda) := \int_0^{2\pi} \sigma(x;\lambda)dx$, satisfies for all $\lambda \in \mathbb{C}$ the gradient relationship (2.32).

3. The generalized Riemann type hydrodynamical equation at N=3: conservation laws, bi-Hamiltonian structure and Lax type representation

Here we proceed to analyzing conservation laws and bi-Hamiltonian structure of the generalized Riemann type equation (1.1) at $N=3$:

$$
\left.\begin{aligned}
u_t &= v - uu_x \\
v_t &= z - uv_x \\
z_t &= -uz_x
\end{aligned}\right\} := K[u,v,z], \tag{3.1}
$$

where $K : \mathcal{M} \to T(\mathcal{M})$ is a suitable vector field on the periodic functional manifold $\mathcal{M} := C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^3)$ and $t \in \mathbb{R}$ is an evolution parameter. The system (3.1) proves also to possess infinite hierarchies of polynomial conservation laws, being suspicious for complete and Lax type

Namely, the following polynomial functionals are conserved with respect to the flow (3.1):

$$
\begin{aligned}
H_n^{(1)} &:= \int_0^{2\pi} dx z^n (v u_x - v_x u - \frac{n+2}{n+1} z), \\
H^{(4)} &:= \int_0^{2\pi} dx[-7 v_x v^2 u + z(6zu + 2v_x u^2 - 3v^2 - 4vuu_x)], \\
H^{(5)} &:= \int_0^{2\pi} dx(z^2 u_x - 2zvv_x), \quad H^{(6)} := \int_0^{2\pi} dx(z_z v^3 + 3z^2 v_x u + z^3), \\
H^{(7)} &:= \int_0^{2\pi} dx(z_x v^3 + 3z^2 v u_x - 3z^3), \\
H^{(8)} &:= \int_0^{2\pi} dx z(6z^2 u + 3z v_x u^2 - 3zv^2 - 4zvu_x - 2v_x v^2 u + 2v^3 u_x), \\
H^{(9)} &:= \int_0^{2\pi} dx[1001 v_x v^4 u + (1092 z^2 u^2 + 364 z v_x u^3 - \\
&\quad -1092 z v^2 u - 728 z v u_x u^2 - 364 v_x v^2 u^2 + 273 v^4 + 728 v^3 u_x u]), \\
H_n^{(2)} &:= \int_0^{2\pi} dx z_x v z^n, \quad H_n^{(3)} := \int_0^{2\pi} dx z_x (v^2 - 2zu)^n,
\end{aligned}
\tag{3.2}
$$

where $n \in \mathbb{Z}_+$. In particular, as $n = 1, 2, ...$, from (3.2) one obtains that

$$
\begin{aligned}
H_0^{(2)} &:= \int_0^{2\pi} dx z_x v, \quad H_1^{(2)} := \int_0^{2\pi} dx z_x z v, ..., \\
H_1^{(3)} &:= \int_0^{2\pi} dx z_x (v^2 - 2uz), \\
H_2^{(3)} &:= \int_0^{2\pi} dx z_x (v^4 + 4z^2 u^2 - 4zv^2 u), ...,
\end{aligned}
\tag{3.3}
$$

and so on.

Making use of the iterative property, similar to that, formulated above in Proposition 2.1, one can construct the following hierarchy of non-polynomial dispersive and dispersionless conservation laws:

$$
\begin{aligned}
H_1^{(1/4)} &= \int_0^{2\pi} dx(-2u_{xx} u_x z_x + u_{xx} v_x^2 + 2u_x^2 z_{xx} - \\
&\quad -u_x v_{xx} v_x + 3v_{xx} z_x - 3v_x z_{xx})^{1/4}, \\
H_2^{(1/3)} &= \int_0^{2\pi} dx(-v_{xx} z_x + v_x z_{xx})^{1/3}, \\
H_3^{(1/3)} &= \int_0^{2\pi} dx(v_{xx} u_x - v_x u_{xx} - z_{xx})^{1/3}, \\
H_1^{(1/2)} &= \int_0^{2\pi} dx[-2v u_x z_x + v_x^2 + z(-u_x v_x + 3z_x)]^{1/2}, \\
H_2^{(1/2)} &= \int_0^{2\pi} dx(8u_x^3 z_x - 3u_x^2 v_x^2 - 18 u_x v_x z_x + 6v_x^3 + 9z_x)^{1/2}, \\
H_1^{(1/5)} &= \int_0^{2\pi} dx(-2u_{xxx} u_x z_x + u_{xxx} v_x^2 + 6u_{xx}^2 z_x - 6u_{xx} u_x z_{xx} - \\
&\quad -3u_{xx} v_{xx} v_x + 2u_x^2 z_{xxx} - u_x v_{xxx} v_x + \\
&\quad +3u_x v_{xx}^2 + 3v_{xxx} z_x - 3v_x z_{xxx})^{1/5}, \\
H_3^{(1/3)} &= \int_0^{2\pi} dx[k_1 u(-v_{xx} z_x + v_x z_{xx}) + k_1 v(u_{xx} z_x - u_x z_{xx}) + \\
&\quad +z(k_2 u_{xx} v_x - k_2 u_x v_{xx} + k_1 z_{xx} + k_2 z_{xx}) + \\
&\quad +k_2(-2v_x u_x z_x + v_x^3 + 3z_x^2)]^{1/3}
\end{aligned}
\tag{3.4}
$$

where $k_j \in \mathbb{R}, j=\overline{1,3}$, are arbitrary real numbers. Below we will attempt to generalize the crucial relationship (2.37) from Section 2 on the case of the Riemann type hydrodynamical system (3.1). Namely, we will assume, based on the Remark (2.3), that there exists its following linearization:

$$D_t^3 f_3(\lambda) = 0, \tag{3.5}$$

modeling the starting generalized Riemann type hydrodynamical equation (1.1) at $N=3$, and where $f_3(\lambda) \in C^\infty(\mathbb{R}^2;\mathbb{C})$ for all values of the parameter $\lambda \in \mathbb{C}$. The scalar equation (3.5) can be easily rewritten as the system of three linear equations

$$D_t f_1 = 0, \quad D_t f_2 = \mu_1 f_1, \quad D_t f_3 = \mu_2 f_2 \tag{3.6}$$

where we have defined a vector $f := (f_1, f_2, f_2)^\intercal \in C^\infty(\mathbb{R}^2;\mathbb{C}^3)$ and naturally introduced constant numbers $\mu_j := \mu_j(\lambda) \in \mathbb{C}, \quad j=\overline{1,2}$. It is easy to observe now that, owing to the former result (1.4), the system of equations (3.6) allows the following solution representation:

$$\begin{aligned} f_1(x,t) &= \tilde{g}_1(u-tv+zt^2/2, v-zt, x-tu+vt^2/2-zt^3/6), \\ f_2(x,t) &= t\mu_1\tilde{g}_1(u-tv+zt^2/2, v-zt, x-tu+vt^2/2-zt^3/6)+ \\ &\quad +\tilde{g}_2(u-tv+zt^2/2, v-zt, x-tu+vt^2/2-zt^3/6), \\ f_3(x,t) &= \mu_1\mu_2\frac{t^2}{2}\tilde{g}_1(u-tv+zt^2/2, v-zt, x-tu+vt^2/2-zt^3/6)+ \\ &\quad +t\mu_2\tilde{g}_2(u-tv+zt^2/2, v-zt, x-tu+vt^2/2-zt^3/6)+ \\ &\quad +\tilde{g}_3(u-tv+zt^2/2, v-zt, x-tu+vt^2/2-zt^3/6), \end{aligned} \tag{3.7}$$

where $\tilde{g}_j \in C^\infty(\mathbb{R}^3;\mathbb{C}), j=\overline{1,3}$, are arbitrary smooth complex valued functions. The system (3.6) transforms into the equivalent vector equation

$$D_t f = q(\mu) f, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \mu_1(\lambda) & 0 & 0 \\ 0 & \mu_2(\lambda) & 0 \end{pmatrix}, \tag{3.8}$$

which should be compatible both with a suitably chosen equation for derivative

$$f_x = \ell[u,v,z;\lambda] f \tag{3.9}$$

with some matrix $\ell[u,v,z;\lambda] \in SL(3;\mathbb{C})$, defined on the functional manifold $\mathcal{M}$, and with the Lie-Lax equation (2.34), rewritten as the following system of equations

$$D_t\varphi = \begin{pmatrix} 0 & v_x & z_x \\ -1 & -u_x & 0 \\ 0 & -1 & -u_x \end{pmatrix}\varphi, \quad D_t = \partial/\partial t + u\partial/\partial x, \tag{3.10}$$

where the vector $\varphi := \varphi(x;\lambda) \in T^*(\mathcal{M})$ is considered as the one factorized by means of a solution $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^3)$ to (3.9), satisfying the identity (2.36). Namely, it is assumed that the following quadratic trace-relationship

$$\varphi = tr(\Phi[\lambda;u,v,z]\ f\otimes f^\intercal) \tag{3.11}$$

holds for some vector valued matrix $\Phi[\lambda;u,v,z] \in \mathbb{E}^3 \otimes End\ \mathbb{E}^3$, defined on the manifold $\mathcal{M}$, where "$\otimes$" means the standard tensor product of vectors from the Euclidean space $\mathbb{E}^3$. Making now use of the determining expressions (2.36), (3.11) and (3.8), one can find by means of some slightly cumbersome but tedious calculations that $\mu_1(\lambda) = \lambda$, $\mu_2(\lambda) = 1$, $\lambda \in \mathbb{C}$, and the matrix representation of the derivative (3.9)

$$\ell[u,v,z;\lambda] = \begin{pmatrix} \lambda u_x & -v_x & z_x \\ 3\lambda^2 & -2\lambda u_x & \lambda v_x \\ 6\lambda^2 r[u,v,z] & -3\lambda & \lambda u_x \end{pmatrix}, \tag{3.12}$$

compatible with determining equation (3.10), where a smooth mapping $r : \mathcal{M} \to \mathbb{R}$ satisfies the differential relationship

The latter possesses a wide set $\mathcal{R}$ of different solutions amongst which there are the following:

$$
r \in \mathcal{R} := \{[(xv - u^2/2)/z]_x, (v_x - u_x^2/6)z_x^{-1}, \frac{u_x^3/3 - u_x v_x + 3z_x/2}{2u_x z_x - v_x^2}, \\
(v_x v^3/6 - u_x v^2 z/2 + u z_x(uz - v^2)/6 + vz^2)z^{-3}\}. \tag{3.14}
$$

Note here that only the third element from the set (3.14) allows the reduction $z = 0$ to the case $N = 2$. Thus, the resulting Lax type representation for the Riemann type dynamical system (3.1) ensues in the form:

$$
f_x = \ell[u, v, z; \lambda]f, \quad f_t = p(\ell)f, \quad p(\ell) := -u\ell[u, v, z; \lambda] + q(\lambda),
$$

$$
\ell[u, v, z; \lambda] = \begin{pmatrix} \lambda u_x & -v_x & z_x \\ 3\lambda^2 & -2\lambda u_x & \lambda v_x \\ 6\lambda^2 r[u, v, z] & -3\lambda & \lambda u_x \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.15}
$$

$$
p(\ell) = \begin{pmatrix} -\lambda u u_x & u v_x & -u z_x \\ -3u\lambda^2 + \lambda & 2\lambda u u_x & -\lambda u v_x \\ -6\lambda^2 r[u, v, z]u & 1 + 3u\lambda & -\lambda u u_x \end{pmatrix},
$$

where $f \in C^\infty(\mathbb{R}^2; \mathbb{C}^3)$ and $\lambda \in \mathbb{C}$ is a spectral parameter.

The next problem, which is of great interest, consists in proving that the generalized hydrodynamical system (3.1) is a completely integrable bi-Hamiltonian flow on the periodic functional manifold $\mathcal{M}$, as it was proved above for the system (2.2).

That dynamical system (3.1) is bi-Hamiltonian that follows easily as a simple corollary from the fact that it possesses the Lax type representation (3.15) and from the general Lie-algebraic integrability theory [17, 18, 21]. Taking into account that dynamical system (3.1) possesses many (at least 4) Lax type representations, one derives that it possesses many (at least 4) different pairs of compatible co-symplectic structures, every of which generates its own infinite hierarchy of commuting to each other conservation laws. Moreover, the involution of conservation laws belonging to different hierarchies fails owing to their non-compatibility. As finding of these structures is adjoint with cumbersome enough analytical calculations, we present below only a one pair of related co-symplectic structures, making use of the standard properties of determining them Lie-Lax equation (2.18).

To tackle with the related task of retrieving the Hamiltonian structure of the dynamical system (3.1), it is enough, as in Section 2, to construct [18, 20] exact non-symmetric solutions to the Lie-Lax equation

$$
\psi_t + K'^{,*}\psi = grad\ \mathcal{L}, \quad \psi' \neq \psi'^{,*}, \tag{3.16}
$$

for some functional $\mathcal{L} \in D(\mathcal{M})$, where $\psi \in T^*(\mathcal{M})$ is, in general, a quasi-local vector, such that the system (3.1) allows the following Hamiltonian representation:

$$
K[u, v, z] = -\eta\ grad\ H[u, v, z], \\
H_\eta = (\psi, K) - \mathcal{L}, \quad \eta^{-1} = \psi' - \psi'^{,*}. \tag{3.17}
$$

As a test solution to (3.16) one can take the one

$$
\psi = (u_x/2, 0, -z_x^{-1}u_x^2/2 + z_x^{-1}v_x)^\intercal, \quad \mathcal{L} = \frac{1}{2}\int_0^{2\pi}(2z + vu_x)dx,
$$

which gives rise to the following co-implectic operator:

$$
\eta^{-1} := \psi' - \psi'^{,*} = \begin{pmatrix} \partial & 0 & -\partial u_x z_x^{-1} \\ 0 & 0 & \partial z_x \\ -u_x z_x^{-1}\partial & z_x\partial & \begin{matrix} \frac{1}{2}(u_x^2 z_x^{-2}\partial + \partial u_x^2 z_x^{-2}) - \\ -(v_x z_x^{-2}\partial + \partial v_x z_x^{-2}) \end{matrix} \end{pmatrix}. \tag{3.18}
$$

This expression is not strictly invertible, as its kernel possesses the translation vector field d/dx : $\mathcal{M} \to T(\mathcal{M})$, with components $(u_x, v_x, z_x)^\intercal \in T(\mathcal{M})$, that is $\eta^{-1}(u_x, v_x, z_x)^\intercal = 0$.

Nonetheless, upon formal inverting the operator expression (3.18), we obtain by means of simple enough, but slightly cumbersome, direct calculations, that the Hamiltonian function equals

$$H_\eta := \int_0^{2\pi} dx(u_x v - z). \tag{3.19}$$

and the implectic η-operator looks as

$$\eta := \left(\begin{array}{ccc} \partial^{-1} & u_x\partial^{-1} & 0 \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x & \partial^{-1}z_x \\ 0 & z_x\partial^{-1} & 0 \end{array} \right). \tag{3.20}$$

The same way, representing the Hamiltonian function (3.19) in the scalar form

$$H_\eta = (\psi, (u_x, v_x, z_x)^\intercal), \quad \psi = \frac{1}{2}(-v,\ u + ..., -\frac{1}{\sqrt{z}}\partial^{-1}\sqrt{z})^\intercal, \tag{3.21}$$

one can construct a second implectic (co-symplectic) operator $\vartheta : T^*(\mathcal{M}) \to T(\mathcal{M})$, looking up to $O(\mu^2)$ terms, as follows:

(3.22)

$$\vartheta = \left(\begin{array}{ccc} \mu(\frac{(u^{(1)})^2}{z^{(1)}}\partial + \partial\frac{(u^{(1)})^2}{z^{(1)}}) & 1 + \frac{2\mu}{3}(\frac{u^{(1)}v^{(1)}}{z^{(1)}}\partial + 2\partial\frac{u^{(1)}v^{(1)}}{z^{(1)}}) & \frac{2\mu}{3}(\partial\frac{(v^{(1)})^2}{z^{(1)}} + \partial u^{(1)}) \\ -1 + \frac{2\mu}{3}(\partial\frac{u^{(1)}v^{(1)}}{z^{(1)}} + 2\frac{u^{(1)}v^{(1)}}{z^{(1)}}\partial) & \begin{array}{c} \frac{2\mu}{3}(\frac{(v^{(1)})^2}{z^{(1)}}\partial + \partial\frac{(v^{(1)})^2}{z^{(1)}}) + \\ +\frac{2\mu}{3}(u^{(1)}\partial + \partial u^{(1)}) \end{array} & 2\mu\partial v^{(1)} \\ \frac{2\mu}{3}(\frac{(v^{(1)})^2}{z^{(1)}}\partial + u^{(1)}\partial) & 2\mu v^{(1)}\partial & \mu(\partial z^{(1)} + z^{(1)}\partial) \end{array} \right) + O(\mu^2),$$

where we put, by definition, $\vartheta^{-1} := (\psi' - \psi'^{,*})$, $u := \mu u^{(1)}, v := \mu v^{(1)}, z := \mu z^{(1)}$ as $\mu \to 0$, and whose exact form needs some additional simple but cumbersome calculations, which will be presented in a work under preparation.

The operator (3.22) satisfies the Hamiltonian vector field condition:

$$(u_x, v_x, z_x)^\intercal = -\vartheta\ grad\ H_\eta, \tag{3.23}$$

following easily from (3.21).

The results obtained above can be formulated as the following proposition.

Proposition 3.1. *The Riemann type hydrodynamical system (1.1) at $N = 3$ is equivalent to a completely integrable bi-Hamiltonian flow on the functional manifold $\mathcal{M}$, allowing the Lax type representation (3.15) and the compatible pair of co-symplectic structures (3.20) and (3.22).*

The infinite hierarchy of conservation laws like (3.4) and related recurrent relationships can be regularly reconstructed, if to compute the asymptotical solutions to the following Lie-Lax equation:

$$\begin{aligned} L_{\tilde{K}}\tilde{\varphi} &= \tilde{\varphi}_\tau + \tilde{K}'^{,*}\tilde{\varphi} = 0, \\ \tilde{\varphi} &\simeq \tilde{a}(x;\lambda)\exp\{\lambda^2\tau + \partial^{-1}\tilde{\sigma}(x;\lambda)\}, \end{aligned} \tag{3.24}$$

where, by definition, $\tilde{a}(x;\lambda) \simeq \sum_{j\in\mathbb{Z}_+}\tilde{a}_j[u,v,z]\lambda^{-j}$, $\tilde{\sigma}(x;\lambda) \simeq \sum_{j\in\mathbb{Z}_+\cup\{-1\}}\tilde{\sigma}_j[u,v,z]\lambda^{-j}$ as $\lambda \to \infty$, and

$$\begin{aligned} \frac{d}{d\tau}(u,v,z)^\intercal \quad &: \quad = -3\eta\ grad\ H_3^{(1/3)}[u,v,z] = \\ &= \left. \begin{array}{c} -(u_x^2h^{-2})_x + v_x^{-1}(v_x^2h^{-2})_x \\ -v_xu_x^{-1}(u_x^2h^{-2})_x + z_x^{-1}(z_x^2h^{-2})_x \\ -z_xu_x^{-1}(z_x^2h^{-2})_x \end{array} \right\} := \tilde{K}[u,v,z], \\ H_3^{(1/3)} \quad &: \quad = \int_0^{2\pi} h[u,v,z]dx, \quad h[u,v,z] = (v_{xx}u_x - u_{xx}v_x - z_{xx})^{1/3}, \end{aligned} \tag{3.25}$$

is a Hamiltonian vector field on the functional manifold $\mathcal{M}$ with respect to a suitable evolution parameter $\tau \in \mathbb{R}$. Since the vector fields (3.25) and (3.1) are commuting to each other on the whole manifold $\mathcal{M}$, the functionals $\tilde{H}_j^{(1/3)} := \int_0^{2\pi}\tilde{\sigma}_j[u,v,z]dx,\ j \in \mathbb{Z}_+\cup\{-1\}$, will be functionally independent conservation laws for both these dynamical systems.

The Lax type integrability of the Riemann type hydrodynamical equation (1.1) at $N = 2$ and $N = 3$, stated above, allows one to speculate that it is also integrable for arbitrary $N \in \mathbb{Z}_+$.

Concerning the evident difference between analytical properties of the cases $N = 2$ and $N = 3$, we can easily observe that it is related with structures of the corresponding Lax type operators (2.38) and (3.15): in the first case the corresponding r-equation (3.13) is trivial (that is empty), but in the second case it is already nontrivial, allowing many different solutions. This situation generalizes, as we will see below, to the case $N \geq 4$, thereby explaining the appearing diversity of related Lax type representations.

To support this hypothesis we will prove below that also at $N = 4$ it is equivalent to a Lax type integrable bi-Hamiltonian dynamical system on the suitable smooth 2π-periodic functional manifold $\mathcal{M} := C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^4)$, possesses infinite hierarchies of polynomial dispersionless and dispersive non-polynomial conservation laws.

4. The generalized Riemann type hydrodynamical equation at N=4: conservation laws, bi-Hamiltonian structure and Lax type representation

The Riemann type hydrodynamical equation (1.1) at $N = 4$ is equivalent to the nonlinear dynamical system

$$\left.\begin{aligned} u_t &= v - uu_x \\ v_t &= w - uv_x \\ w_t &= z - uw_x \\ z_t &= -uz_x \end{aligned}\right\} := K[u, v, w, z], \tag{4.1}$$

where $K : \mathcal{M} \to T(\mathcal{M})$ is a suitable vector field on the smooth 2π-periodic functional manifold $\mathcal{M} := C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^4)$. To state its Hamiltonian structure, we need to find an exact non-symmetric functional solution $\psi \in T^*(\mathcal{M})$ to the Lie-Lax equation (3.16):

$$\psi_t + K^{',*}\psi = grad\ \mathcal{L} \tag{4.2}$$

for some smooth functional $\mathcal{L} \in D(\mathcal{M})$, where

$$K' = \begin{pmatrix} -\partial u & 1 & 0 & 0 \\ -v_x & -u\partial & 1 & 0 \\ -w_x & 0 & -u\partial & 1 \\ -z_x & 0 & 0 & -u\partial \end{pmatrix}, \quad K^{',*} = \begin{pmatrix} u\partial & -v_x & -w_x & -z_x \\ 1 & \partial u & 0 & 0 \\ 0 & 1 & \partial u & 0 \\ 0 & 0 & 1 & \partial u \end{pmatrix} \tag{4.3}$$

are, respectively, the Frechet derivative of the mapping $K : \mathcal{M} \to T(\mathcal{M})$ and its conjugate. The small parameter method [18], applied to equation (4.2), gives rise to the following its exact solution:

$$\begin{aligned} \psi &= (-w_x, v_x/2, 0, -\frac{v_x^2}{2z_x} + \frac{u_x w_x}{z_x})^\intercal, \\ \mathcal{L} &= \int_0^{2\pi} (zu_x - vw_x/2)dx. \end{aligned} \tag{4.4}$$

As a result, we obtain right away from (4.2) that dynamical system (4.1) is a Hamiltonian system on the functional manifold $\mathcal{M}$, that is

$$K = -\vartheta\ grad\ H, \tag{4.5}$$

where the Hamiltonian functional equals

$$H := (\psi, K) - \mathcal{L} = \int_0^{2\pi} (uz_x - vw_x)dx \tag{4.6}$$

and the co-implectic operator equals

$$\vartheta^{-1} := \psi' - \psi^{',*} = \begin{pmatrix} 0 & 0 & -\partial & \partial\frac{w_x}{z_x} \\ 0 & -u\partial & 0 & -\partial\frac{v_x}{z_x} \\ -\partial & 0 & 0 & \partial\frac{u_x}{z_x} \\ \frac{w_x}{z_x}\partial & -\frac{v_x}{z_x}\partial & \frac{u_x}{z_x}\partial & \begin{matrix}\frac{1}{2}[z_x^{-2}(v_x^2 - 2u_xw_x)\partial + \\ +\partial(v_x^2 - 2u_xw_x)z_x^{-2}]\end{matrix} \end{pmatrix}. \tag{4.7}$$

The latter is degenerate: the relationship $\vartheta^{-1}(u_x, v_x, w_x, z_x)^\intercal = 0$ holds exactly on the whole manifold $\mathcal{M}$, but the inverse to (4.7) exists and can be calculated analytically

To state the Lax type integrability of Hamiltonian system (4.1) we will apply to it, as in Section 3 above, the standard gradient-holonomic scheme of [18, 20] and find the following its linearization:

$$D_t^4 f_4(\lambda) = 0, \tag{4.8}$$

where $f_4(\lambda) \in C^\infty(\mathbb{R}^2;\mathbb{C})$ for all $\lambda \in \mathbb{C}$. Having rewritten (4.8) in the form of the linear system

$$D_t f = q(\lambda) f, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix} \tag{4.9}$$

with $\lambda \in \mathbb{C}$ being an arbitrary constant, for the vector $f \in C^\infty(\mathbb{R}^2;\mathbb{C}^4)$ one obtains easily, owing to the relationships (1.4), the following functional representation:

$$\begin{aligned}
f_1(x,t) &= \tilde{g}_1(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt, \\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!), \\
f_2(x,t) &= t\lambda\tilde{g}_1(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt, \\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!) + \\
&\quad +\tilde{g}_2(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt, \\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!), \\
f_3(x,t) &= \lambda^2\frac{t^2}{2}\tilde{g}_1(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt, \\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!) + \\
&\quad +t\lambda\tilde{g}_2(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt, \\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!) + \\
&\quad +\tilde{g}_3(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt, \\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!), \\
f_4(x,t) &= \lambda^3\frac{t^3}{3!}\tilde{g}_1(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt, \\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!) + \\
&\quad +\lambda^2\frac{t^2}{2}\tilde{g}_2(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt, \\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!) + \\
&\quad +t\lambda\tilde{g}_3(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt, \\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!) + \\
&\quad +\tilde{g}_4(u - tv + wt^2/2 - xt^3/3!, v - wt + zt^2/2, w - zt, \\
&\quad x - tu + vt^2/2 - wt^3/3! + zt^4/4!),
\end{aligned} \tag{4.10}$$

where $\tilde{g}_j \in C^\infty(\mathbb{R}^4;\mathbb{C}), j = \overline{1,4}$, are arbitrary smooth complex valued functions.

Based now on the expressions (4.9) and (4.10), one can construct the related Lax type representation for dynamical system (4.1) in the following compatible form:

$$f_x = \ell[u,v,z,w;\lambda]f, \quad f_t = p(\ell)f, \quad p(\ell) := -u\ell[u,v,w,z;\lambda] + q(\lambda), \tag{4.11}$$

where

$$\ell[u,v,w,z;\lambda] := \begin{pmatrix} -\lambda^3 u_x & \lambda^2 v_x & -\lambda w_x & z_x \\ -4\lambda^4 & 3\lambda^3 u_x & -2\lambda^2 v_x & \lambda w_x \\ -10\lambda^5 r_1 & 6\lambda^4 & -3\lambda^3 u_x & \lambda^2 v_x \\ -20\lambda^6 r_2 & 10\lambda^5 r_1 & -4\lambda^4 & \lambda^3 u_x \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix},$$

$$p(\ell) = \begin{pmatrix} \lambda u u_x & -\lambda^2 u v_x & \lambda u w_x & -u z_x \\ \lambda + 4\lambda^4 u & -3\lambda^3 u u_x & 2\lambda^2 u v_x & -\lambda u w_x \\ 10\lambda^5 u r_1 & \lambda - 6\lambda^4 u & 3\lambda^3 u u_x & -\lambda^2 u v_x \\ 20\lambda^6 \ldots & 10\lambda^5 \ldots & \lambda + 4\lambda^4 \ldots & \lambda^3 \ldots \end{pmatrix}, \tag{4.12}$$

the mappings $r_j : \mathcal{M} \to \mathbb{R}, j = \overline{1,2}$, satisfy the functional-differential equations

$$D_t r_1 + r_1 D_x u = 1, \qquad D_t r_2 + r_2 D_x u = r_1, \tag{4.13}$$

similar to (3.13), considered already above, thereby being a Lax type integrable dynamical system on the functional manifold $\mathcal{M}$.

The equations (4.13), as it is easy to demonstrate [14, 11] by means of standard differential-algebraic methods, possess a lot of different solutions, amongst which there are functional expressions:

$$\begin{aligned} r_1 &= D_x\left(\frac{uw^2}{2z^2} - \frac{vw^3}{3z^3} + \frac{vw^4}{24z^4} + \frac{7w^5}{120z^4} - \frac{w^6}{144z^5}\right), \\ r_2 &= D_x\left(\frac{uw^3}{3z^3} - \frac{vw^4}{6z^4} + \frac{3w^6}{80z^5} + \frac{vw^5}{120z^5} - \frac{w^7}{420z^6}\right). \end{aligned} \tag{4.14}$$

Owing to the existence of the Lax type representation (4.11), (4.12) and the related gradient like relationship (2.32), we can easily derive that the Hamiltonian system (4.1) is also simultaneously a bi-Hamiltonian flow on the functional manifold $\mathcal{M}$. And as it was mentioned before, it possesses many bi-Hamiltonian structures, depending on a chosen solution to the corresponding functional-differential equations (4.13).

In addition, we can construct, making use of the results above and the approach of Section 1, the infinite hierarchies of related conservation laws for (4.1), both dispersionless polynomial and dispersive non-polynomial ones:

a) polynomial conservation laws:

$$\begin{aligned} H^{(9)} &= \int_0^{2\pi} dx(vw_x - uz_x), \qquad H^{(19)} = \int_0^{2\pi} dx z_x(w^2 - 2vz), \\ H^{(16)} &= \int_0^{2\pi} dx(3u_x z^2 + 4w_x vz + 2z_x vw), \quad H^{(20)} = \int_0^{2\pi} dx(z_x w - zw_x), \\ H^{(17)} &= \int_0^{2\pi} dx[3u_x z(3uz + 2vw) - 6v_x z(uw + v^2) + \\ &\quad +6w_x(uvz + 2uw^2 - v^2 w) + z(w^2 - 2vz)], \\ H^{(18)} &= \int_0^{2\pi} dx[k_1(z_x(2uw - v^2) + z^2) + k_2((2w_x(uz - vw) + 2z_x(v^2 - uw))], \end{aligned} \tag{4.15}$$

b) non-polynomial conservation laws:

(4.16)

$$\begin{aligned} H^{(10)} &= \int_0^{2\pi} dx(w_x^2 - 2v_x z_x)^{1/2}, \\ H^{(11)} &= \int_0^{2\pi} dx\Big(u_{xx}z_x - u_x z_{xx} + v_x w_{xx} - v_{xx}w_{xx}\Big)^{\frac{1}{3}}, \\ H^{(12)} &= \int_0^{2\pi} dx\Big(9u_x^2 z_x - 6u_x v_x w_x + 2v_x^3 - 12 v_x z_x + 6w_x^2\Big)^{\frac{1}{3}}, \\ H^{(13)} &= \int_0^{2\pi} dx\Big(u(2v_x z_x - w_x^2) + v(v_x w_x - 3u_x z_x) + \\ &\quad +w(u_x w_x - v_x^2 + 2z_x) + z(u_x v_x - 2w_x)\Big)^{\frac{1}{3}}], \end{aligned}$$

$$H^{(14)} = k_1 H^{(14)} + k_2 H^{(14)} + H^{(14)}, \quad H^{(15)} = k_1 H^{(15)} + k_2 H^{(15)} + H^{(15)}$$

where

$$
\begin{aligned}
H_1^{(14)} &= \int_0^{2\pi} \Big(u_{xx}(2v_x z_z - w_x^2) + v_{xx}(v_x w_x - 3u_x z_x) + \\
&+ w_{xx}(u_x w_x - v_x^2 + 2z_x) + z_{xx}(u_x v_x - 2w_x) \Big)^{\frac{1}{4}}, \\
H_2^{(14)} &= \int_0^{2\pi} dx \Big(z_x w_{xx} - z_{xx} w_x \Big)^{\frac{1}{3}}, \\
H_3^{(14)} &= \int_0^{2\pi} dx [k_1 (v(2v_x z_x - w_x^2) + z(4z_x - u_x w_x) + w(v_x w_x - 3u_x z_x)) + \\
&+ k_2 z(2z_x + v_x^2 - u_x w_x)]^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
H_1^{(15)} &= \int_0^{2\pi} dx [u_{xxx}(2v_x z_x - w_x^2) + v_{xxx}(v_x w_w - 3u_x z_x) + \\
&+ z_{xxx}(u_x v_x - 2w_x) + w_{xxx}(u_x w_x - v_x^2 + 2z_x) + \\
&+ 3u_{xx}(v_{xx} z_x - 3v_x z_{xx} + w_{xx} w_x) + 3v_{xx}(2u_x z_{xx} - \\
&- v_{xx} w_x + v_x w_{xx}) - 3w_{xx}^2 u_x]^{\frac{1}{5}}, \\
H_2^{(15)} &= \int_0^{2\pi} dx \Big(4u_x^2 w_x^2 - 4u_x v_x^2 w_x - 8u_x z_x w_x + v_x^4 + +4v_x^2 z_x + 4z_x^2 \Big)^{\frac{1}{4}}, \\
H_3^{(15)} &= \int_0^{2\pi} dx \, \{k_3[u(z_x w_{xx} - z_{xx} w_x) + \\
&+ v(v_x z_{xx} - v_{xx} z_x) + z z_{xx} + w(u_{xx} z_x - u_x z_{xx})] + \\
&+ k_4[z(u_{xx} w_x - u_x w_{xx} + 2z_{xx}) + w(u_{xx} z_x - u_x z_{xx} - v_{xx} w_x + v_x w_{xx})] + \\
&+ k_5 z_x (v_x^2 - 2u_x w_x + 2z_x)\}^{\frac{1}{3}},
\end{aligned} \tag{4.17}
$$

and $k_j \in \mathbb{R}$, $j = \overline{1,5}$, are arbitrary constants. We observe also that the Hamiltonian functional (4.6) coincides exactly up the sign with the polynomial conservation law $H^{(9)} \in D(\mathcal{M})$.

Remark 4.1. It is worth here to remark [14] that the generalized Riemann type hydrodynamical equation (1.1) can be once more naturally generalized to the following also integrable Riemann type equation

$$D_t^N u = 0, \quad D_t := \partial/\partial x + a(\hat{u})\partial/\partial t, \tag{4.18}$$

where $N \in \mathbb{Z}_+$ and $a \in C^\infty(\mathcal{M};\mathbb{R})$ is an arbitrary smooth mapping. The corresponding to (4.18) nonlinear dynamical system

$$
\begin{aligned}
u_t^{(0)} &= u^{(1)} - a(\hat{u}) u_x^{(1)}, \\
u_t^{(1)} &= u^{(2)} - a(\hat{u}) u_x^{(2)}, \\
& \ldots\ldots \\
u_t^{(N-2)} &= u^{(N-1)} - a(\hat{u}) u_x^{(N-2)} \\
u_t^{(N-1)} &= -a(\hat{u}) u_x^{(N-1)}.
\end{aligned} \tag{4.19}
$$

will be also a bi-Hamiltonian Lax type integrable dynamical system on the phase space $\mathcal{M}$.

Thereby, the calculations above ensue the formulation of the following proposition.

Proposition 4.2. *The Riemann type hydrodynamical system (1.1) at $N = 4$ is equivalent to a completely integrable bi-Hamiltonian flow on the functional manifold $\mathcal{M}$, allowing the Lax type representation (4.11) and whose co-implectic structure is given by expression (4.7).*

Concerning the general case $N \in \mathbb{Z}_+$, applying successively either the symplectic approach

the Riemann type hydrodynamical system (1.1) and (4.18) the infinite hierarchies of dispersive and dispersionless conservation laws, co-symplectic structures and related Lax type representations, what is a topic of the next work under preparation.

5. Conclusion

As follows from the results obtained in this work, the generalized Riemann type hydrodynamical equation (1.1) possesses many infinite hierarchies of conservation laws, both dispersive nonpolynomial and dispersionless polynomial. This fact can be easily explained by the fact that the corresponding dynamical system (1.2) allows many, plausibly, infinite set of algebraically independent compatible implectic structures, which generate via the standard gradient like relationship (2.30) the related infinite hierarchies of conservation laws, and as a by-product, infinite hierarchies of the associated Lax type representations. Such a situation within the theory of Lax type integrable nonlinear dynamical systems meets, virtually, for the first time and may appear to be interesting from different point of view, as well as theoretical and practical. Keeping in mind these and some other important aspects of the generalized Riemann type hydrodynamical equation (1.1), we consider that they deserve additional thorough investigation in the future.

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